Risk or Regulatory Capital? Bringing distributions back in the foreground

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Abstract

This paper discusses the regulatory requirement (Basel Committee, ECB-SSM and EBA) to measure financial institutions' major risks, for instance Market, Credit and Operational, regarding the choice of the risk measures, the choice of the distributions used to model them and the level of confidence. We highlight and illustrate the paradoxes and the issues observed implementing an approach over another and the inconsistencies between the methodologies suggested and the goal to achieve. This paper make some recommendations to the supervisor and proposes alternative procedures to measure the risks.[2]

Key words: Risk measures - Sub-additivity - Level of confidence - Extreme value distributions - Financial regulation

[1]Disclaimer: The opinions, ideas and approaches expressed or presented are those of the authors and do not necessarily reflect Santander's position. As a result, Santander cannot be held responsible for them.

[2]This paper has been written in a very particular period of time as most regulatory papers written in the past 20 years are currently being questioned by both practitioners and regulators themselves. Some distress or disarray has been observed among risk managers as most models required by the regulation were not consistent with their own objective of risk management. The enlightenment brought by this paper is based on an academic analysis of the issues engendered by some pieces of regulation and it has not for purpose to create any sort of polemic.

1 Introduction

The ECB-SSM [3], the EBA [4] and the Basel Committee are currently reviewing the methodological framework of risk modelling. In this paper, we analyse some of the issues observed measuring the risks as prescribed that would be useful to address in the future regulatory documents.

1.1 Problematic

During the current crisis, the failure of models and the lack of capture of extreme exposures led regulators to change the way risks were measured either by requiring financial institutions to use particular families of distributions (Gaussian (BCBS (2005)), sub-exponantial (EBA (2014*b*))), either by changing the way dependencies were captured (EBA (2014*b*)) or suggesting switching from the VaR (Value-at-Risk)[5] to sub-additive risk measures like the ES (Expected Shorfall)[6] (BCBS (2013)). Indeed, risk modelling had played a major role during the crisis which began in 2008 either as catalysts or triggers. The latest changes proposed by the authorities have been motivated by the will to come closer to the reality of the financial markets.

Before capturing dependencies, the choice of the probability distributions used to model the risks and their associated measures are key points for practitioners and regulators. From a technical point of view, it is now accepted that the most relevant piece of information for risk managers is contained in the tails of the distributions characterising the risk factors they are willing to control. Thus it appears sensible that regulators, following theoretical and empirical studies and evolution of the risks associated to markets, financial products and actors behaviours adjust

[3]European Central Bank - Single Supervisor Mechanism

[4]European Banking Authority

[5]Given a confidence level $p \in [0,1]$, the VaR associated to a random variable X is given by the smallest number x such that the probability that X exceeds x is not larger than $(1-p)$

$$VaR_{(1-p)\%} = \inf(x \in \mathbb{R} : P(X > x) \leq (1-p)). \tag{1.1}$$

[6]For a given p in $[0,1]$, η the $VaR_{(1-p)\%}$, and X a random variable which represents losses during a prespecified period (such as a day, a week, or some other chosen time period) then,

$$ES_{(1-p)\%} = E(X|X > \eta). \tag{1.2}$$

their requirements. Nevertheless analysing in details these requests, we note - inside the guidelines - some confusions and misleading interpretations which cannot help to robustly evaluate and control these risks in financial institutions and also do not permit constructive exchanges between regulators and practitioners in order to reach the stability objective of the financial industry.

Thus, the purpose of this paper is to discuss some parts of the methodological framework proposed for risk modelling by the regulators and its evolution from 1995 to 2015, focusing on their strong incentive to use: (i) specific distributions to characterise the risks, (ii) specific risk measures, (iii) specific associated confidence level, and to apply these strategies independently from each others. We illustrate in the following that distributions, risk measures and confidence levels are three facets of a single object and are therefore indivisible. Thus, we argue that the approaches proposed by the regulators in the guidelines focusing on risk modelling engender a bias (positive or negative) in the computation of the risks, and consequently a distortion in the corresponding capital requirements, as soon as the problem of the measurement is not being dealt with in its globality.

Some of the following points are mainly addressed in this paper:

1. Is the choice of a particular risk measure ensures conservativeness?
2. What is the impact of the choice of a particular distribution on the associated risk measure?
3. For a single kind of risk: given a risk measure, what choice of the confidence level p is really appropriate?
4. When we use a VaR_p measure, for which distributions is the sub-additivity[7] property fulfiled as soon as we consider several risk factors?

[7] A coherent risk measure is a function $\rho : \mathcal{L}^\infty \to \mathbb{R}$:

- Monotonicity:If $X_1, X_2 \in \mathcal{L}$ and $X_1 \leq X_2$ then $\rho(X_1) \leq \rho(X_2)$
- Sub-additivity: If $X_1, X_2 \in \mathcal{L}$ then $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$
- Positive homogeneity: If $\lambda \geq 0$ and $X \in \mathcal{L}$ then $\rho(\lambda X) = \lambda\rho(X)$
- Translation invariance: $\forall k \in \mathbb{R}$, $\rho(X + k) = \rho(X) - k$

5. Given that each risk type is modelled considering different distributions and using different p-s, how can the sub-additivity criterion be fulfilled?

1.2 Questions raised by Regulatory proposals

For three categories of risks (market, credit[8] and operational), regulatory proposals regarding the choice of the distributions, the risk measures and the associated confidence level p are analysed. Issues related to their implementation have been highlighted in the following.

1.2.1 Market risks

1 - In the BCBS (1995) document, it is explicitly written:"The Committee has examined carefully how banks' value-at-risk measures based on the parameters described above can be converted into a capital requirement that appropriately reflects the prudential concerns of supervisors. One of the problems of recognising banks' value-at-risk measures as an appropriate capital charge is that the assessments are based on historical data and that, even under a 99% confidence interval, extreme market conditions are excluded. The Committee does not believe that a ten-day value-at-risk measure provides sufficient comfort for the measurement of capital for a number of reasons, which include: the past is not always a good guide to the future; the assumptions about statistical "normality" built into some models may not be justified, i.e. there may be "fat tails" in the distribution curve; the correlations assumed in the model may prove to be incorrect; market liquidity may become inadequate to close out positions."

These proposals suggest several remarks concerning various very different concepts.

- First, the regulator says that the choice of the VaR as the risk measure excludes to take into account extreme events. This statement is not correct as the choice of the VaR is not the issue, it is the choice of the underlying distribution with which the associated quantile is evaluated that determines if the extreme events are captured or not. This question actually implied a second question about what is an extreme event as answering this question would suppose a complete information set.
- Second, the regulator discusses the inadequacy of using a "ten-day value at risk...." for the

[8] Credit Value Adjustment and Wrong Way Risk Included (BCBS (2011a))

measurement of the capital. Indeed, it is generally an error to use "the ten-day value at risk" but the reasons evoked by the regulators are sources of confusion. The errors found their origins in two main issues, the choice of the model by the practitioners (independence between the risk factors) on the first hand, and the implied Gaussian behaviour of the risk models, on the other hand. It would be more appropriate from a practical point of view to invite practitioners to work in a more robust way investigating both the properties and the patterns of their data: (i) Are they independent or not? (ii) Whatever the answer to this first question, what is the most appropriate model (according to adequacy, conservativeness, or some other criteria)? (iii) How to capture the dependencies (not only between two risk factors using the concept of correlation which assumes also that there is some linearity between the risk factors, but a more general dependence architecture)? (iv) What is "the" distribution characterising the data set (it can be Gaussian, fat tailed, thin tailed, asymmetric, symmetric, multimodal etc....)? (v) How to justify the choice of the probability distribution they retain?

- Third, the question of the choice of the information set is crucial, but saying that the use of historical data is "not a good way to work" is particularly dangerous. Indeed, the historical data set is *the only original information set available* for the modeler. Concerning this information set, a more relevant question is to decide the period the modeler has to use and the length of this period. Thus, there is a huge mis-understanding from the regulator concerning the information set the practitioners have to use : "good" data sets do not exist.

- Fourth, when regulators said that modellers need to capture market liquidity with the information set, it was only rhetorical. As soon as a market is illiquid it creates a systemic risk which is another problem, consequently the question of the definition of liquidity risk should be raised. Why did the regulator introduce this issue in 1995? A debate is largely opened on the concept and even in 2015, to our knowledge, it does not emerge a "correct" and useful definition for this kind of risk in the literature, as the liquidity coverage ratio (LCR) supposed to evaluate the buffer covering the risk of being illiquid converges to zero as soon as the market tends to dry up. Note that we will not address this issue in the following as it is out of scope of this paper, but it was worth mentioning it with respect to the question of the data set quality.

2 - In Interpretive Issues with respect to the revisions to the market risk framework in November 2011, Page 8, paragraph 718 (xciii) (BCBS (2011*b*)) regarding the incremental risk capital, the Basel Committee stipulates that "in combination with the relevant rules on IRC, it is accepted that banks model issuer interdependence assuming multivariate normal distributions or normal copula (e.g. between asset values, credit spreads or default times) or must they show that such model assumptions do not underestimate risk?".

This proposal is surprising and very restrictive looking at the modelling of dependencies between the risk factors. Indeed the Gaussian copula takes the same information in both tails (it does not consider that asymmetry between information sets can exist). Furthermore, it does not consider the importance of fat tails, which is not consistent with the behaviour of most risks observed on financial markets (especially during the crisis). Since 2011, a lot of research using copula (like the Archimeadean copula for instance) and vines which take asymmetrical behavior and information in the tails has been conducted. This remark, is interesting as a Gaussian copula structure correspond to the inverse of a multivariate Gaussian distribution. We see that once again the world seems Gaussian regulatory speaking. We are quickly illustrating the issue below, though we will see that it is not helping addressing issues related to risk measures.

3 - Nevertheless the regulator in the same document states (page 9): "The onus is on the bank to justify the modelling choices and their impact to the national supervisor. Normal distributions or normal copula may not be assumed uncritically. The impact of such modelling choices must be analysed in the validation."

One may wonder why regulators propose so restrictive models (Gaussian framework in the previous paragraph) to say after that they may not be good enough? Why not just giving the possibility to financial institutions to model the risk factors in the way they consider to be the best fit, and then rely on an independent validation process to validate their choice. [9]

4 - In the Consultative Document concerning the Fundamental review of the trading book (BCBS (2013)) "A revised market risk framework", October 2013, Page 3, the Basel Committee proposes

[9] We recall that the idea of a floor on some metrics (capital, etc.) has been mentioned and may also be relevant.

to move from Value-at-Risk (VaR) to Expected Shortfall (ES) as "a number of weaknesses have been identified with using VaR for determining regulatory capital requirements, including its inability to capture “tail risk”". The ES measures the riskiness of a position by considering both the size and the likelihood of losses above a certain confidence level. The Committee has agreed to use a 97.5th ES for the internal models-based approach and has also used that approach to calibrate capital requirements under the revised market risk standardised approach.

With this more recent document it appears that the regulator thinks that the use of the ES instead of the VaR risk would permit to capture the most relevant information to measure the risks. This is not necessarily true as it still depends on the choice of the distributions used for the computation of this ES. Nevertheless, we know that this last measure is more interesting than the VaR as considering the same distribution it provides a better information concerning the amplitude of the risk, but if the fitted distribution is inappropriate the problem of capture of the extreme events remains the same. Besides, the choice of the level of confidence, for instance 97.5 is also arbitrary (this point has been illustrated in the next Section). Why does the regulator move from 99 % (in 1995) to 97,5 % (in 2012)? - Why do not they propose 95% or another value p?.

1.2.2 Credit Risk and Counterparty Credit Risk

1 - In the International Convergence of Capital Measurement and Capital Standards, November 2005, Page 60 recommendation 71 (BCBS (2005)), the Basel Committee explicitly discusses the evaluation of the credit risk using an underlying Gaussian distribution along other parameters.

This proposal remains very limited and a dangerous approach, even if some robust works can be done when practitioners compute the probability of default (PD) (Guégan et al. (2013)), the loss given default (LGD) and the exposure of default (EAD). Using a Gaussian distribution even shifted considering the other parameters implies that the essence of the modelled risk has a Gaussian behaviour. Indeed, the intensity of default is transformed into a Gaussian quantile and somehow compared to the 99th percentile of the same Gaussian distribution. This is highly questionable and overly simplistic. What led to that conclusion? How is that possible to be backtested? This approach is nothing more than a transformed VaR. To the regulators' credit this model did not lead (yet) to massive failures, though low default portfolio credit risk modelling is currently being highly questioned.

Another issue that is worth mentioning but not dealt with in this paper is the independence between PD, LGD and EAD. This is not consitent with the piece of regulation requiering banks to capture wrong way risk, i.e. the upper tail dependence between PD and EAD (BCBS (2011*a*)).

2 - In the Article 383(4) of Regulation 575/2013 European Banking Authority - Consultation Paper - On Draft Regulatory Technical Standards (RTS) on credit valuation adjustment risk for the determination of a proxy spread and the specification of a limited number of smaller portfolios, and in the Article 383 of Regulation (EU) 575/2013 Capital Requirements Regulation - CRR, July 2013, Page 7 (EBA (2014*a*)), it is said "that institutions are permitted (i) to use a VaR model for the measurement of specific market risk of debt instruments, (ii) to use an internal Expected Positive Exposure (EPE) model for the calculation of the exposure values to counterparty credit risk on the majority of their business, but use other methods (Mark-to-Market Method, Standardised Method or Original Exposures Method) for smaller portfolios".

It appears that these proposals are problematic as some "norms" are defined for banks internal models even if these models are totally inadequate when we fit them to real data. Nowhere regulators or experts from regulatory institutions provide the assumptions which justify the use of such models or such methodology. How can we interpret this absence of justifications? Is that a political decision? If the answer is yes, then what are the objectives? It seems to be the will of a "one size fits all" approach to be able to compare banks risk management performance. Unfortunately, if the model is inadequate, then the quality of the risk management will not be properly reflected and therefore the outcomes of the model misleading for both the authorities and the practitioners. Besides, it appears to be a bit despotic and to transfer the burden from the regulator onto the banks.

The credit value adjustment (CVA) is closely related to the evaluation of derivatives. The assumption is that models used to evaluate derivatives such as Black & Scholes as these models do not capture the risk of a counterparty defaulting. The CVA in its nature challenges the use of Gaussian like distributions as it questions the soundness of the risk neutral valuation. Indeed the risk neutral valuation does not capture the risk of a counterparty to default, therefore an adjustment has been created to add a buffer in capital. The expected positive exposure (EPE) used to evaluate the CVA is quite interesting too, as a symmetrical distribution such as the Gaussian distribution is used to capture an asymetric exposure (positive) (Gregory (2012)). This does not seem very sensible. It seems that once again, the regulation implies the use of a particular distribution even if it is not appropriate except maybe to simplify the calculations.

1.2.3 Operational risks

1 - In their article 312 of Regulation (EU) No 575/2013, June 2014, Page 43, the European Banking Authority - Consultation Paper - Draft Regulatory Technical Standards on assessment methodologies for the Advanced Measurement Approaches for operational risk (EBA (2014*b*)) states that "the competent authority shall verify that an institution pays particular attention to the positive skewness and leptokurtosis of the data when selecting a severity distribution. When the data are much dispersed in the tail, empirical curves shall not be used to estimate the tail region. Sub-exponential distributions shall be used for this purpose unless there exist exceptional reasons to apply other functions, which shall be in any case properly addressed and fully justified to prevent undue reduction of the capital figures." In Pages 17-18, "sub-exponential distributions" are defined as distributions whose right tail decreases slower than the exponential distribution. The class of sub-exponential distributions includes the lognormal, log-gamma, log-logistic, generalised Pareto, Burr, and Weibull (with shape parameter < 1) (Guégan and Hassani (2014)). The Weibull (with shape parameter > 1) and gamma distributions do not belong to the class of Sub-exponential distributions. Sub-exponential distributions can better represent the shape of the data in the tail (other than their skewness in the body) by allowing estimates of parameters that do not depend on the higher order statistical moments".

Comparing the proposal for operational risk modelling (Guégan and Hassani (2009), Guégan and Hassani (2013*b*)) with proposals to model other types of risks, we observe that for operational risks, experts thinking is ahead, however, some questions should be raised.

- First, why is it forbidden to use "empirical curves" to fit the distributions? This notion which is not specified, probably refers to the non-parametric fit which can be used, avoiding the price to pay with analytical forms. It is a pity not inciting practitioners to use this technique because it is well known by modelers that non-parametric fittings when they are correctly conducted, provides better fits than any parametric distributions and is generally used as a benchmark.

- Second, it is difficult to understand - from a parametric point of view - why the regulators focus on the limited list of distributions and why they consider them "at the same level" knowing that they have specific behaviors which cannot take into account all the features of the risks, and also knowing that the methodologies to fit them are so different. On the other hand, this list cannot be exhaustive as there exist other classes of distributions (quite common) useful to fit these kinds of risks and the regulatory paper does not make any reference to them assuming that a good fit would be found anyway using the distributions enumerated. The approach is in our opinion far too restrictive.

- As it seems that there is a mis-understanding regarding the foundation of their proposals concerning this list of distributions, we provide here an alternative and complementary approach for the choice of the distributions the modelers can use. If we consider the class of sub-exponential distributions and in particular those listed in the document we can say that from a probabilistic and statistical point of view, these distributions have very different properties. The generalised Pareto distribution belongs to the class of extreme value distribution and is fitted on data sets selected above a certain threshold. The Weibull distribution belongs to the class of extreme value distributions through the Theorem of Fisher-Tippett (Fisher and Tippett (1928)) appearing as the max -distribution for a certain class of distributions. In practice we will fit this distribution on data sets built from the original data using block maxima method selecting the maxima inside the original data set. The other distributions, if they present any interest in risk modelling (what about the Burr distribution?) will be fitted on the whole sample. In fine, there is a lot of confusion for the choice of these distributions. It would be more interesting to introduce and classify correctly the classes of distributions they propose for their use in practice. Indeed, a large panel of classes of distributions can be considered: the Generalised Hyperbolic (GH) Class of distributions (Barndorff-Nielsen (1977)), the α-stable distributions (Samorodnitsky and

Taqqu (1994)), the g-and h distributions (Hoaglin (1985)) from one hand, and the extreme value distributions (Weibull, Fréchet, Gumbel), and the GPD distributions on the other hand. It is important to make a distinction between these two classes of distributions because the former ones are fitted on the whole sample and the latter ones on some specific sub-samples which is fundamental in terms of risk management. Note also that the techniques of estimations differ for all these classes of distributions. Indeed, we use the whole sample (original data set) to estimate the GH, α-stable and g-and-h distributions, and maximum likelihood procedures will be used; Hill and Pickand methods are considered for estimating the parameters of the GPD distributions, and maximum likelihood method associated with block maxima methodology is used for the extreme value classes of distributions. The choice of the distributions cannot be split from the difficulty to estimate its parameters and the underlying information set. For instance the GPD distribution is very difficult to fit on nearly all data set because of the estimation of the threshold which is a key parameter for this class of distributions and whose estimate is generally very unstable. An error in its estimation can create error, distortion and confusion on the allocation of the capital. It is surprising that the regulators do not take into account these issues before imposing such distributions. Their proposal is worthy of a Prévert setting but, unfortunately, does not correspond to a robust approach.

2 - In the same document (page 17, item 24), regulators discuss the choice of risk measure:"risk measure means a single statistic extracted from the aggregated loss distribution at the desired confidence level, such as Value at Risk (VaR), or shortfall measures (e.g. Expected Shortfall, Median Shortfall)".

This definition is particularly limitative. How the risk measures computed for different factors with different levels can be aggregated? Does the regulator has in mind the use of a spectral measure, then how can we use it? This would be interesting but the concept has never been discussed in any regulatory document. Thus, how robust is the method proposed?

1.3 To summarise

While these documents are addressing the main issues, we believe that some documents are too prescriptive, preventing banks from going beyond the proposals and focusing more on the capital calculations than the risk management itself. Regarding the calculation of the capital requirement from the knowledge of the risk factors, the main points concerns the choice of the distribution, the choice of the risk measure and the choice of p. The regulator would like to impose some choices. In the previous subsection these ones and the strategies to evaluate the risks independently from each other are questioned. Consequently, in the following we discuss the distributions suggested in the regulatory documents to model the risks and we analyse the soundness of the risk measures and *a priori* confidence levels associated to these ones.

2 Alternative strategies to the regulatory papers

In the previous Section, the methodological choices implied by the regulation have been presented and discussed, focusing on the nature of the distributions used to characterise a risk, the type of risk measure and the dependence structure to be applied. We point out some confusion and mis-understanding concerning the proposals of the regulators for these very technical points which are fundamental for the risk management of a banking institution as soon as these choices are determinant in the computation of the capital requirements, i.e. these should be risk sensitive. Thus, in this Section, using some data sets we illustrate and highlight the impact of these choices on practitioners[10] perceptions of a risk.

2.1 Data set and strategy

We have selected a data set provided by a Tier European bank representing "Execution, Delivery and Process Management" risks from 2009 to 2013. "Execution, Delivery and Process Management" risk is a sub-category of operational risk. This data set is characterized by a distribution right skewed (positive skewness) and leptokurtic.

In order to follow regulators' requirements in their different guidelines, we choose to fit on this data set some of the distributions proposed inside the regulatory documents and also others which seem more appropriate regarding the properties of the data set. We retain seven distribu-

[10] Academics, Risk Managers or Authorities

tions. They are estimated (i) on the whole sample: the lognormal distribution (asymmetric and medium tailed), the Weibull distribution (asymmetric and thin tailed), a Generalised Hyperbolic (GH) distribution (symetric or asymmetric, fat tailed on an infinite support), an Alpha-Stable distribution (symmetric, fat tailed on an infinite support), a Generalised Extreme value (GEV) distribution (asymmetric and fat tailed), (ii) on an adequate subset: the Generalised Pareto (GPD) distribution (asymmetric, fat tailed) calibrated on a set built over a threshold, a Generalised Extreme value (GEVbm) distribution (asymmetric and fat tailed) fitted using maxima coming from the original set. The whole data set contains 98082 data points, the sub-sample used to fit the GPD contains 2943 data points and the sub-sample used to fit the GEV using the block maxima approach contains 3924 data points. The objective of these choices is to evaluate the impact of the selected distributions on the risk representation, i.e. how the initial empirical exposures are captured and transformed by the model.

Table 1 exhibits parameters' estimates for each distribution selected[11]. The parameters are estimated by maximum likelihood, except for the GPD which implied a POT (Guégan et al. (2011)) approach and the GEV fitted on the maxima of the data set (maxima obtained using a block maxima method (Gnedenko (1943))). The quality of the adjustment is measured using both the Kolmogorov-Smirnov and the Anderson-Darling tests. The results presented in Table 1 show that none of the distribution is adequate. This is usually the case when fitting unimodal distributions to multimodal data set. Indeed, multimodality of the distributions is a frequent issue modelling operational risks as the risk categories combine multiple kinds of incident, for instance, a category combining external fraud will contain the fraud card on the body, commercial paper fraud in the middle, cyber attack and Ponzi scheme in the tail. This comes back to an issue we have discussed previously, where empirical distributions would be more appropriate than fitted analytical distributions as it may help capturing multimodality. Unfortunately this solution has been crossed out by regulators as this solution is considered not being able to capture the tail properly. The use of fitted analytical distributions has been preferred despite the fact that sometimes no proper fit can be found and the combination of multiple distributions may lead to a high number of parameters and consequently to even more unstable results. Nevertheless

[11]In order not to overload the table the standard deviation of the parameters are not exhibited but are available upon request.

the question of multi-modality becomes more and more important concerning the fitting of any data set. In this paper we do not discuss this issue in more details as it is out of the scope of our objective as the regulators never suggest this approach: we provide some discussion on this methodological aspect of risk modelling in the conclusion (some references are for instance, Wang (2000) or Guégan and Hassani (2015a)).

Using this data set and the fit of the seven distributions, we compute for each distribution the associated VaR_p and ES_p for different values of p: the results are provided in Table 3. Then, in order to consider the question of sub-additivity and the competition between VaR_p and ES_p when we have more than one risk factor we show the impact of the choice of the distributions on the computation respectively of the $\text{VaR}_p(X+Y)$ and $\text{VaR}_p(X)$ + $\text{VaR}_p(Y)$ where X and Y are two risk factors. In this exercise, we compute the previous quantities in the following way:

- As $\text{VaR}_p(X)$ is a quantile, $p \in [0,1]$, we can build the whole spectrum of the VaR, i.e. the inverse of the cumulative distribution function. Summing $\text{VaR}_p(X)$ and $\text{VaR}_p(Y)$ for each value of p provides us with $\text{VaR}_p(X)$ + $\text{VaR}_p(Y)$.

- To obtain $\text{VaR}_p(X+Y)$, another approach is adopted. In a first step we randomly generate X and Y using the distribution fitted previously. Then X and Y are aggregated. The resulting cumulative distribution function is built and its inverse provide the spectrum of $\text{VaR}_p(X+Y)$.

The results are provided in the Tables 4 to 15. We also illustrate these last results by graphes: they are given in Figures 1 to 7. We analyse now the results of these tables and Figures.

2.2 VaR

In this Section we analyse the results presented in Table 3 which provides the values obtained for the VaR_p and the ES_p computed from the seven distributions fitted on the data set or some sub-samples, and also of Tables 4 to 15 which permit to address the property of non sub-additivity associated to the VaR. We also discuss figures 1 to 7 which illustrate this notion for the VaR, combining different distributions.

First and foremost, from Table 2, we can see that, p given, the choice of the distribution has a tremendous impact on the value of the risk measure, i.e. if a lognormal distribution is used, then the 99% VaR is equal to 3 039, while it is 22 for a Weibul adjusted on the same data and 84 522 using a GPD. A corollary is that the 90% VaR of the GPD is much higher than all the 99% VaR calculated with any other distribution. A first conclusion would be, what is the point of imposing a percentile if the practitioners are free to use any distribution (c.f. Operational Risk AMA)?

Second, given p, between the four distributions fitted on the whole sample (for instance lognormal, Weibull, GH and alpha-stable), compared to the GPD fitted above a threshold, we observe a huge difference for the VaR. A change in the threshold may either give higher or lower values depending, therefore the VaR is highly sensitive to the value of the threshold. Now, comparing the result obtained on the GEV fitted on the whole sample to the GEV fitted on the bock maxima, we observe the one focusing on the tails (GEV block maxima) leads to lower VaRs than a GEV capturing the entire set of information. Therefore, a more conservative information set associated to the appropriate distribution may lead to a lower VaR. The number of blocks considered may have some impact but nothing comparable to the effect of the threshold on a GPD. The Weibull which is a distribution contained in the GEV provides the lowest VaR.

Table 3 exhibits the risk measures obtained from fully correlated random variables. It is interesting to note that the risk measures obtained on fully correlated random variables and the sum of the risk measures obtained univariately are really similar. This means that as soon as we sum the VaR obtained on two variables we mechanically assume an upper tailed correlation for the random variables. Therefore, besides of being conservative the sum of univariate VaRs taken at the same level prevents the capture of any diversification benefit. Fully correlated random variables do not embed any diversification benefit by definition. Consequently, the analysis regarding the sub-additivity of the risk measures have to be performed in another way.

So, as presented earlier, we randomly generated values from the distribution fitted before and combined them two by two. By way of this process we generated some random correlations and as a matter of fact some diversification. Then, we compared the risk measures obtained from the combination of random variables and the sum of the risk measures computed on the random variables taken independently. Figures 1 to 5 represents the spectra of these two items and consequently allows comparing the risk measure obtained for different confidence levels. These figures show that depending on the combinations of distributions, the VaR may always be sub-additive, never sub-additive, only sub-additive in the tails, only sub-additive in the body or may be more erratic, i.e. can be sub-additive initially, then become non-sub-additive, and finally become sub-additive again in the tails. These observations are supported by Tables 12 to 14. Tables 14 also show the impact of the discretisation of the distributions on the risk measures, as this impact the sub-additivity. This last property is not only associated to the choice of the risk measure but definitively to the choice of the distribution and of the confidence level?

Analysing the results in details, we see in Table 4, for p fixed, that the VaR is never sub-additive if the lognormal distribution is associated to a GPD, while if the lognormal distribution is associated to any of the others, the VaR is usually sub-additive in the tails but not at the end of the body part. Note that if the lognormal is associated with an identical lognormal, the differences we have observed are only due to numerical errors related to the sampling. We expect the two values to be absolutely identical. Though, it is interesting to note that the random generation of numbers can be the root cause of non sub-additive results. Identical analysis can be done on other combinations (see table 6). Looking at Table 6 it appears that when the GPD has a positive location parameter, this prevents any combination from being sub-additive, because by construction the 0th percentile of the GPD is equal to the location parameter which should be according to Pickand's theorem (Pickands (1975)), sufficiently high. At the 95th percentile, the VaR is always sub-additive as soon as a lognormal distribution is involved except if it is combined with a GPD. For the other distributions, it is not always true. For example, the VaR obtained combining a Weibull and a GEV fitted on the whole sample is not sub-additive. Table 7 shows that the use of an Alpha-Stable combined with any other distribution except the GPD provides sub-additive risk measures at the 99% level.

In Parallel, Figures 1 to 5 allows a more discriminating analysis of the behaviour of the component $VaR_p(X+Y)$ versus $VaR_p(X)+VaR_p(Y)$. In Figure 1, we show that the sub-additivity property is only verified for high percentile when we use a combination of a Weibull and a GH distributions, i.e. for $p > 90\%$. Besides, the gap tends to enlarge as the percentiles increase.

Figure 2 exhibits a non sub-additive VaR from the 95th percentile, when we use the combination of an Alpha-Stable distribution and a GEV fitted with the block maxima method, but the differences are not as large as on Figure 1. Figure 3 shows that combining two identical distributions does not always produce sub-additive risk measures though it should always be the case: this can be due to numerical errors engendered by the random generation of data points and the discretisation of the distribution. On Figure 4 and 5 we observe that the VaRs obtained from the combination of an Alpha-Stable distribution and a GH distribution or an Alpha-stable distribution and a GEV calibrated on maxima are never sub-additive below 70% . For comparison purposes, Figures 6 and 7 illustrate that the combination of two elliptical distributions (respectively the Gaussian and the Student-t distributions) always leads to sub-additive VaRs.

2.3 Expected Shortfall

In this Section the results of the Table 3 are analysed. It provides the values obtained for the VaR_p and the ES_p computed from the seven distributions fitted on the data set or some sub-samples, and also of the Tables 4 to 15 which permit to verify the sub-addivity property of the ES risk measure.

The ES calculations are linked to the distribution used to model the underlying risks. Looking at Table 2, at the 95%, we observe that the ES goes from 18 for the Weibull to 216 127 for the GPD. Therefore, depending on the distribution used to model the same risk, at the same p level, the ES obtained is completely different. The corollary of that issue is that the ES obtained for a given distribution at a lower percentile will be higher than the ES computed on another distribution at a higher percentile. For example, Table 2 show that the 90% ES obtained from an Alpha-Stable distribution is much higher than the 99.9% ES computed on a lognormal distribution.

The comments regarding the impact of the choice of the information sets on the calculation of the considered risk measure are identical to those stated in the third paragraph of the previous section except regarding 2 points. First, results obtained from the GPD and the alpha-stable distribution are of the same order. Second, the differences between the GPD and the GEV fitted on the block maxima are huge, illustrating the fact that despite being two extreme value distributions, the information captured is quite different.

Regarding the sub-additivity issue, by building the ES always lead to sub-additive values, contrary to the VaR for which this property is not always verified and depends on the underlying distribution. It is interesting to note that if we combine two ES taken at two different levels of confidence p, the ES may not be sub-additive anymore. This is a point that the regulator does not discuss when he says that we have to aggregate the risk measures. This issue is particularly important for risk managers, as soon as the level of confidence prescribed in the regulation guidelines is different from a risk factor to another and appears totally arbitrary.

While the use of several confidence levels $p_i, i = 1, \cdots, k$ allowing to have a spectral representation of the risk measure (VaR or ES) could be interesting but the approach proposed by the regulator which mixes distribution and confidence level is questionable. Indeed, the 70% ES of some combinations may lead to much higher value that the 99.9th (Table 9, WE-GPD vs WE-GH).

2.4 VaR vs Expected Shortfall

Previously we illustrate the fact that, depending on the distribution used and the confidence level chosen, the values provided by VaR_p can be bigger than the values derived for an ES_p and conversely. Thus a question arises: What should we use the VaR or the Expected Shortfall? To answer to this question we can consider several points:

- Conservativeness: Regarding that point, the choice of the risk measure is only relevant for a given distribution, i.e. for any given distribution the VaR_p will always be inferior to the ES_p (assuming only positive values) for a given p. But, if the distribution used to characterise the risk is to be chosen and fitted, then it may happen that for a given level p, the VaR_p obtained from a distribution is superior to the ES_p. For example Table 2 shows that the 99.9% VaR obtained using the GEV distribution fitted with block maxima is superior to the ES obtained for any other distribution at the same level p.

- Sub-additivity: In that case only the expected shortfall guarantees always the sub-additivity of the measure as soon as the p is set, but for some distributions the VaR can be also sub-additive.

- Distribution and p impacts: Table 2 shows that potentially a 90% level ES obtained on a given distribution is larger than a 99.9% VaR obtained on another distribution, e.g. the ES obtained from a GH distribution at 90% is higher that the VaR obtained form a lognormal distribution at 97.5%. Thus is it always pertinent to use a high value for p?

- Parameterisation and estimation: the impact of the calibration of the estimates of the parameters is not negligible (Guégan et al. (2011)), mainly when we fit a GPD. Indeed in that latter case, due to the instability of the estimates for the threshold, the practitioners can largely overfit the risks. Thus, why the regulators still impose this distribution?

3 Conclusion and Recommendations

In the introduction, analysing several guidelines issued by the EBA and the Basel Committee, we pointed out the fact that the regulators impose specific distributions, risk measures and confidence levels to analyse the risk factors in order to evaluate the capital requirements of financial institutions. It appears that their approach is non holistic and their analysis of the risks relies on a disconnection between the choice of the distributions, the risk measures and the confidence level, tools necessary for risks assessments.

In this paper we show that the risk measurement for financial institutions depends intrinsically on how the tools are chosen, i.e. the distribution, the combinations of these distributions, the type of risk measure and the level of confidence. Therefore, the existence of a risk measure as discussed in the regulation is questionable, as for example modifying the level of confidence by a few percents would result in completely different interpretations. The regulators fail to propose an appropriate approach to measure these risks in financial institutions as soon as they do not take into account the problem of risk modelling in its globality.

Regulators are by far too prescriptive and their choices questionable:

- Imposing distributions makes no sense whatever the risks to be modeled. Where are these a priori coming from?

- The regulation reflect some misunderstanding of distributions' properties (probabilist approach) and of the particular properties surrounding their fitting (statistical approach).

- The levels of confidence p seems rather arbitrary. They neither take into account the flexibility of risk measures nor the impact of the underlying distribution, misleading risk managers.

While these fundamental problems are not addressed, others are completely ignored such as the concept of spectral analysis, or of distortion risk measures (Sereda et al. (2010), Guégan and Hassani (2015*a*)). Despite the cosmetic changes included in Basel II and III, the propositions do not enable a better risk management, and banks response to regulatory points are not appropriate as they do not correspond to the reality. It is therefore not surprising that capital calculations and stress testing are still unclear, and that these are not able to capture asymmetric chocs corresponding an extreme incident (black swan, dinosaur or dragon).

Some other questions should also be addressed:

- Is that more efficient in terms of risk management to measure the risk and then build a capital buffer or to adjust the risk taken considering the capital we have? In other word, maybe should banks start optimising their income generation with respect to the capital they already have.

- The previous points are all based on unimodal parametric distributions to characterise each risk factor, what is the impact of using multimodal distributions in terms of risk measurement and management? We believe that an empirical evaluation of the risk provides bank with a reliable benchmark and a starting point in term of what would be an acceptable capital charge or risk assessment.

- One of the biggest issue lies in the fact that we do not know how to combine or aggregate $VaR(X)_{p_1}$, $VaR(Y)_{p_2}$ and $VaR(Z)_{p_3}$ evaluated on three different kinds of risks at three different confidence level p_1, p_2, p_3. This mechanically prevents banks from building a holistic approach from a capital point of view. How should we proceed to solve the problem, should we use $p = max(p_1, p_2, p_3)$, or the min or the average?

- While in this paper we focused on each factor taken independently, the question of the dependence is quite important too. Maybe not as important as the impact of the distribution selected for the risk factor (Guégan and Hassani (2013*a*)) but non addressing this issue properly could lead to a mis-interpretation of the results. The choice of the copula has a direct impact on the dependence structure we would like to apply and the capture of shocks. For instance, a Gaussian or Student t-copula is symmetric, despite the fact that a t-copula with a low number of degrees of freedom could capture tail dependencies, these would not capture asymmetric shocks. Archimedean or Extrema Value copulas associated to a vines strategy would be more appropriate (Guégan and Maugis (2010)).

- In a situation such as depicted by a stress-testing process with forward looking perspective, if the risks are not correctly measured then the foundations will be very fragile and the outcome of the exercise not reliable. Indeed, stressing a situation requires an appropriate initial assessment of the real exposure, otherwise the stress would merely model what should have been captured originally and therefore be useless (Bensoussan et al. (2015), Guégan and Hassani (2015*b*), Hassani (2015)).

We came up to the conclusion that the debate related to the selection of a risk measure over another is not really relevant, and considering issues raised in the previous sections our main recommendation would be to leave as much flexibility as possible to the modellers to build the most appropriate models for risk management purposes initially and then extended with conservative buffers for capital purposes. The idea would be to bring the idea that a good risk management would mechanically limit the exposure and the losses and therefore ultimately reduce the regulatory capital burden. Models should only be a reflexion of the underlying risk framework and not a tool to justify a reduced capital charge. We would like to see more the supervisory face of the authorities and less their regulatory one, in other words we would like them to stop focusing so much on banks risk measurement comparability and more on financial institutions risk understanding. It would probably be wise if both regulators and risk managers were working together (e.g., academic formation open to both corpus, regular workshops, etc., (Guégan (2009))) instead as opponents in order to reach their objective of stability of the financial system for the first and profitability for the second.

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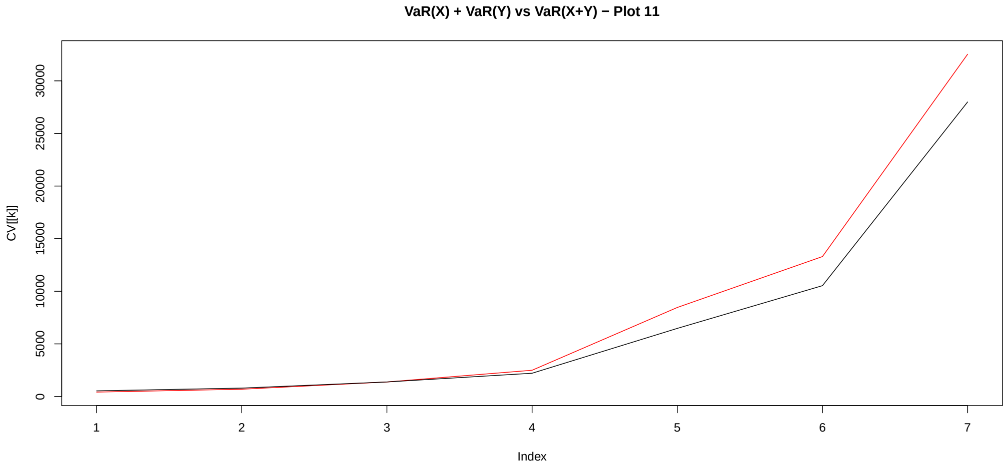

Figure 1: This plot represents the sum of VaR(X) and VaR(Y) (red) versus VaR(X + Y) (black). The random variable X has been generated using a Weibull distribution and Y has been obtained from a Generalised Hyberbolic distribution. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. For high percentiles, the VaR seems to be sub-additive.

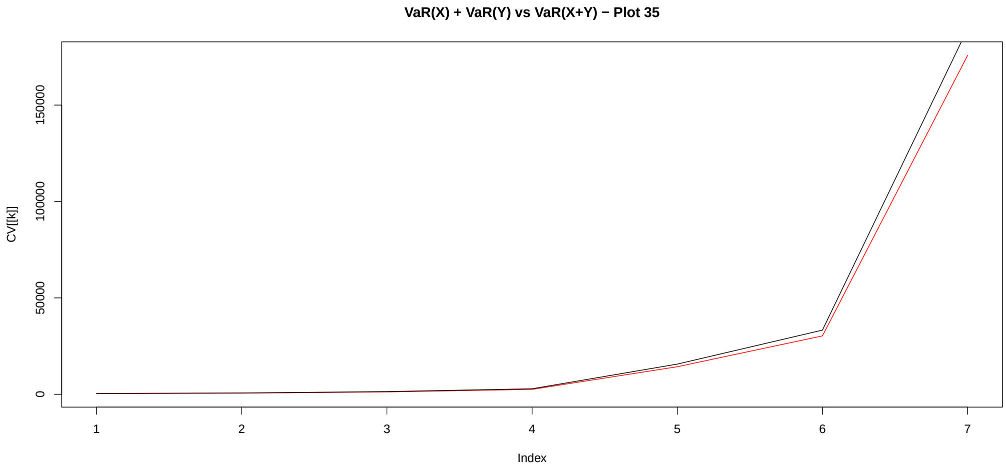

Figure 2: This plot represents the sum of VaR(X) and VaR(Y) (red) versus VaR(X + Y) (black). The random variable X has been generated using a Alpha-stable distribution and Y has been obtained from a GEV distribution calibrated on maxima. The percentile represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. For high percentiles, the VaR is not sub-additive.

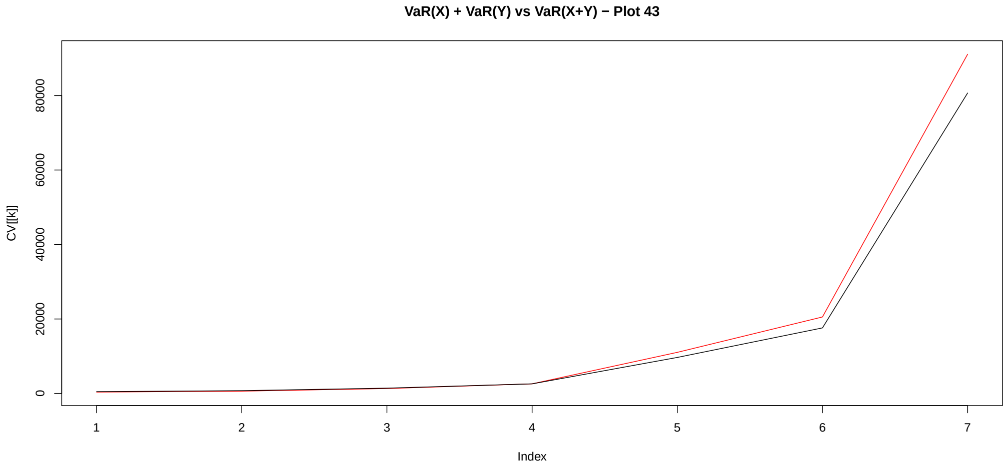

Figure 3: This plot represents the sum of VaR(X) and VaR(Y) (red) versus VaR(X + Y) (black). The random variables X and Y have been obtained from two identical GEV distributions. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th. For high percentiles, the VaR is sub-additive.

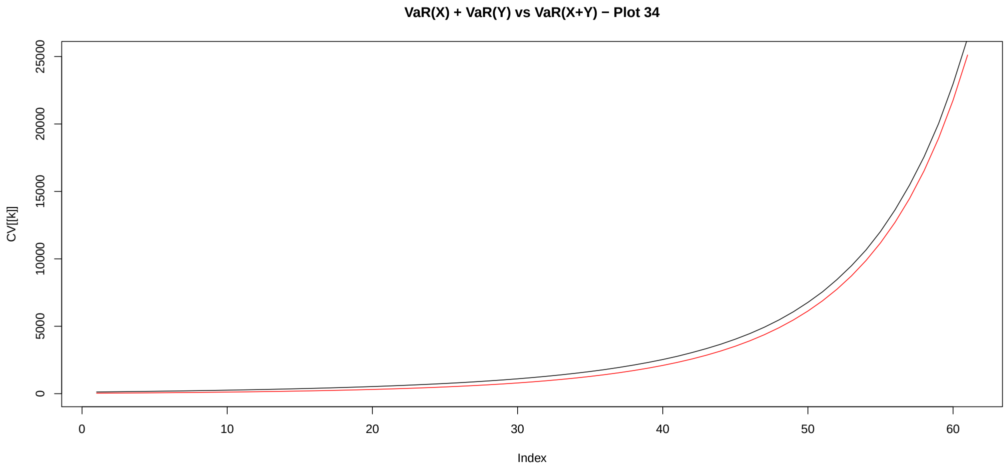

Figure 4: This plot represents the sum of VaR(X) and VaR(Y) (red) versus VaR(X + Y) (black). The random variable X has been generated using a Alpha-stable distribution and Y has been obtained from a Generalised Hyperbolic distribution. The percentiles represented are sequentially going from the 10th to the 70th with a step of 1% between two points. The VaR represented are never sub-additive.

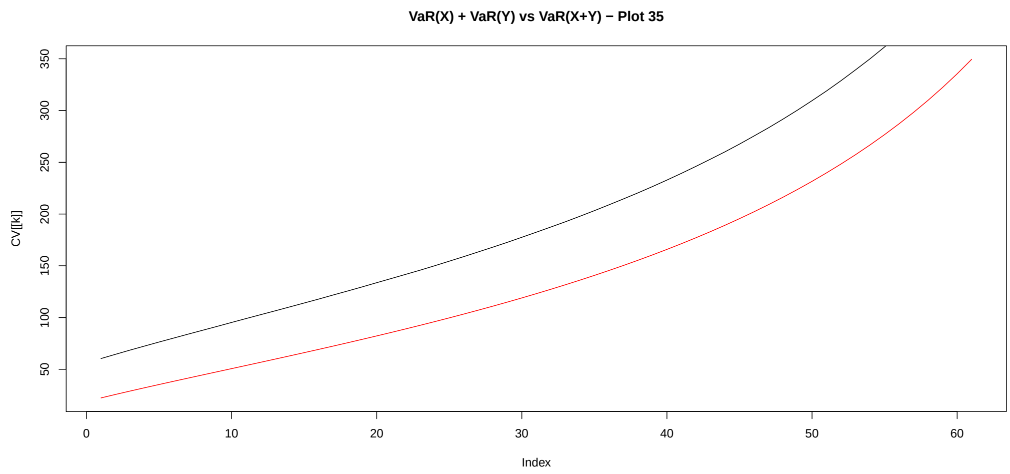

Figure 5: This plot represents the sum of VaR(X) and VaR(Y) (red) versus VaR(X + Y) (black). The random variable X has been generated using a Alpha-stable distribution and Y has been obtained from a GEV distribution calibrated on maxima. The percentiles represented are sequentially going from the 10th to the 70th with a step of 1% between two points. The VaR represented are never sub-additive.

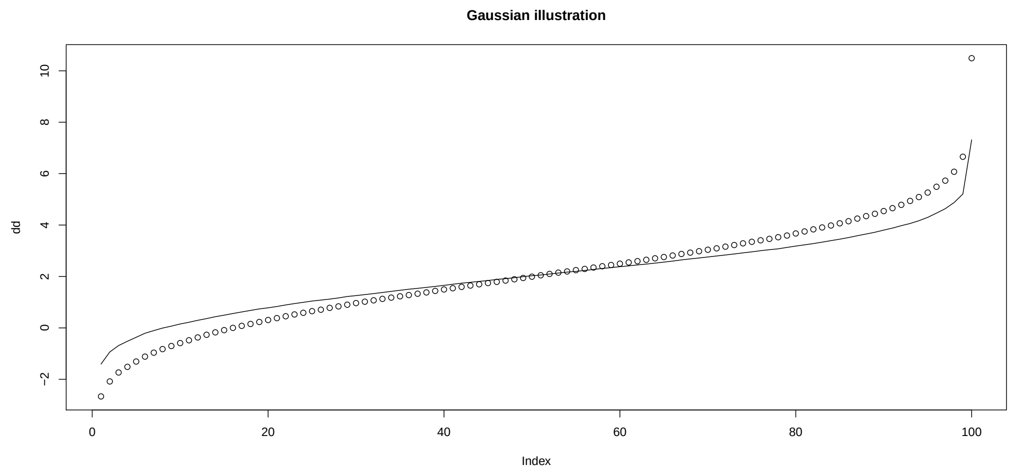

Figure 6: This plot represents the sum of VaR(X) and VaR(Y) (doted line) versus VaR(X + Y) (solid line). The random variable X has been generated using a Gaussian distribution (0, 1) and Y has been obtained from a Gaussian distribution (2, 1). The VaRs represented are always sub-additive.

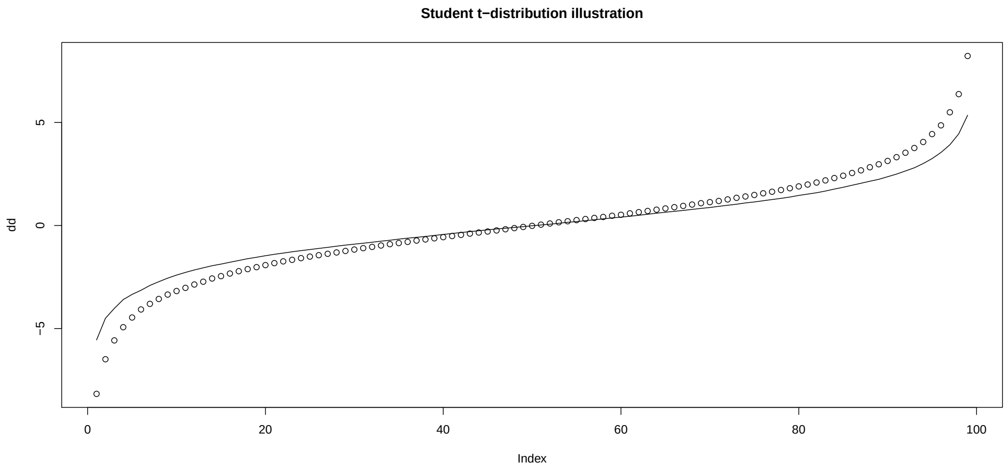

Figure 7: This plot represents the sum of VaR(X) and VaR(Y) (doted line) versus VaR(X + Y) (solid line). The random variable X has been generated using a Student-t distribution (3 df) and Y has been obtained from a Student-t distribution (4 df). The VaRs represented are always sub-additive.

Parameters \ Distribution	LogNormal	Weibull	GPD	GH	Alpha-Stable	GEV	GEV BM
μ	4.412992	-	1541.558	-5.5846599	-	587.855749	50.272385
σ	1.653022	-	-	-	-	2137.940297	64.720971
p	-	0.5895611	-	0.1906536	0.86700	-	-
β	-	182.9008432	1185.8083087	0.1906304	0.95000	-	-
ξ	-	-	0.9039617	-	675 923	3.634536	1.030459
δ	-	-	-	22.5118547	58.18019	-	-
λ	-	-	-	-0.871847		-	-
γ	-	-	-	-	54.21489	-	-
KS	$<$ 2.2e-16	$<$ 2.2e-16	$<$ 2.2e-16	$<$ 2.2e-16	$<$ 2.2e-16	$<$ 2.2e-16	$<$ 2.2e-16
AD	6.117e-09	6.117e-09	6.117e-09	NA	6.117e-09 (d)	6.117e-09	6.117e-09

Table 1: Distribution Parameters - This table presents the parameters obtain for the model. The p-values of both Kolmogorov-Smirnov and Anderson-Darling tests are also provided.

Distribution / %tile	LogNormal		Weibull		GPD		GH		Alpha-Stable		GEV		GEV BM	
	VaR	ES	VaR	ES	VaR	ES	VaR	ES	VaR	ES	VaR	ES	VaR	ES
90%	686	2075	7	13	10 745	142 444	624	2 927	582	50 323	2 097 291	3.072299e+19	626	8 755
95%	1251	3237	11	18	19 906	216 127	1 307	4 961	1 209	99 827	28 699 710	6.144597e+19	1 328	16 617
97.5%	2107	4875	15	23	37 048	351 202	2 601	8 100	2 563	197 937	373 552 400	1.228919e+20	2 762	31 362
99%	3860	8039	22	31	84 522	715 347	5 916	14 522	7 105	488 689	1.073230e10	3.072299e+20	7 177	71 949
99.9%	13646	24435	44	55	675 923	4 893 789	26 435	44 727	98 341	4 691 025	4.702884e+13	3.072299e+21	77 463	550 299

Table 2: Univariate Risk Measures - This table exhibits the VaRs and ESs for the seven types of distributions considered for five confidence level (for instance, 90%, 95%, 97.5%, 99% and 99.9%)

X_1 \ X_2		LogNormal		Weibull		GPD		GH		Alpha-Stable		GEV		GEV BM	
		VaR_{X_1,X_2}	ES_{X_1,X_2}	VaR_{X_1,X_2}	ES_{X_1,X_2}	VaR_{X_1,X_2}	ES_{X_1,X_2}	VaR_{X_1,X_2}	ES_{X_1,X_2}	VaR_{X_1,X_2}	ES_{X_1,X_2}	VaR_{X_1,X_2}	ES_{X_1,X_2}	VaR_{X_1,X_2}	ES_{X_1,X_2}
LogNormal	p_1	2 493	6 103	2 328	4 820	75 546	217 916	2 618	7 753	2 552	44 407	2.834332e+07	1.721696e+21	2 658	19 225
	p_2	4 087	9 057	3 528	6 805	93 442	353 270	4 662	12 064	4 731	85 397	3.665483e+08	3.443392e+21	4 872	34 914
	p_3	7 252	14 693	5 735	10 419	142 038	716 560	9 291	20 539	10 725	203 193	1.072611e+10	8.608479e+21	10 723	76 829
	p_4	24 250	43 132	16 710	27 468	746 560	4 871 970	36 925	59 361	111 991	1 789 756	5.390204e+13	8.608479e+22	83 960	551 875
Weibull	p_1	-	-	2 230	3 637	75 270	219 140	2 424	6 507	2 345	80 360	2.874992e+07	1.648438e+21	2 459	32 126
	p_2	-	-	3 110	4 663	92 818	356 117	4 019	9 939	4 029	157 714	3.746901e+08	3.296876e+21	4 202	61 102
	p_3	-	-	4 425	6 176	140 307	725 572	7 651	16 774	9 003	385 667	1.062522e+10	8.242191e+21	8 856	143 962
	p_4	-	-	8 542	10 777	744 266	4 982 719	29 929	48 745	102 987	3 644 010	4.051415e+13	8.242191e+22	81 979	1 249 637
GPD	p_1	-	-	-	-	150 276	481 590	75 882	198 596	76 003	253 293	2.900050e+07	8.175899e+21	2.852749e+07	8.991075e+20
	p_2	-	-	-	-	185 210	799 161	93 953	314 173	94 929	423 142	3.804500e+08	1.635180e+22	3.719388e+08	1.798215e+21
	p_3	-	-	-	-	281 725	1 667 960	143 016	618 297	149 135	886523	1.055955e+10	4.087950e+22	1.038451e+10	4.495538e+21
	p_4	-	-	-	-	1 482 342	12 136 572	735 168	3 911 547	869 960	6 369 873	5.299107e+13	4.087950e+23	4.773943e+13	4.495538e+22
GH	p_1	-	-	-	-	-	-	2 784	9 385	2 705	73 357	2.880134e+07	3.149731e+20	2.875817e+07	6.426626e+19
	p_2	-	-	-	-	-	-	5 338	14 981	5 470	142 939	3.771517e+08	6.299462e+20	3.674012e+08	1.285325e+20
	p_3	-	-	-	-	-	-	11 469	25 984	13 280	344 980	1.092553e+10	1.574866e+21	1.088133e+10	3.213313e+20
	p_4	-	-	-	-	-	-	47 282	75 037	120 486	3 167 054	5.156340e+13	1.574866e+22	4.991674e+13	3.213313e+21
Alpha-Stable	p_1	-	-	-	-	-	-	-	-	2 649	146 535	2.868921e+07	2.877576e+22	2 822	31 932
	p_2	-	-	-	-	-	-	-	-	5 648	289 288	3.667449e+08	5.755152e+22	5 644	59 930
	p_3	-	-	-	-	-	-	-	-	15 890	709 490	1.007457e+10	1.438788e+23	13 339	137 177
	p_4	-	-	-	-	-	-	-	-	225 543	6 659 012	3.907140e+13	1.438788e+24	96 356	1 107 042
GEV	p_1	-	-	-	-	-	-	-	-	-	-	1.822300e+08	1.096423e+28	2.875582e+07	1.635820e+19
	p_2	-	-	-	-	-	-	-	-	-	-	2.615063e+09	2.192846e+28	3.640431e+08	3.271639e+19
	p_3	-	-	-	-	-	-	-	-	-	-	8.028848e+10	5.482116e+28	1.075443e+10	8.179098e+19
	p_4	-	-	-	-	-	-	-	-	-	-	4.437624e+14	5.482116e+29	4.820065e+13	8.179098e+20
GEV BM	p_1	-	-	-	-	-	-	-	-	-	-	-	-	2 894	59 150
	p_2	-	-	-	-	-	-	-	-	-	-	-	-	5 985	114 221
	p_3	-	-	-	-	-	-	-	-	-	-	-	-	15 597	271 596
	p_4	-	-	-	-	-	-	-	-	-	-	-	-	170 339	2 336 019

Table 3: Correlated Risk Measures - this table presents the VaRs and the ESs obtained on fully correlated random variables.

LN-LN 1	393	663	1,373	2,503	7,721	11,661	27,292
LN-LN 2	395	667	1,376	2,503	7,721	11,677	27,517
LN-WE 1	447	742	1,439	2,427	6,299	8,924	18,498
LN-WE 2	564	826	1,374	2,068	4,654	6,406	14,066
LN-GPD 1	4,321	6,181	11,432	21,158	88,382	163,788	689,569
LN-GPD 2	58,968	60,766	65,759	74,945	138,510	209,859	726,643
LN-GH 1	364	611	1,313	2,569	9,882	16,037	41,329
LN-GH 2	480	742	1,418	2,528	8,205	12,765	30,592
LN-AS 1	377	614	1,269	2,461	10,965	21,402	111,987
LN-AS 2	476	725	1,374	2,472	9,657	18,319	101,929
LN-GV 1	25,132	137,464	2,097,977	28,700,959	$10.73e9$	$134.51e9$	$47,029e9$
LN-GV 2	25,313	138,221	2,095,098	29,156,891	$10.47e9$	$135.38e9$	$45,501e9$
LN-GVb 1	366	614	1,312	2,579	11,037	20,542	91,109
LN-GVb 2	481	742	1,423	2,571	9,670	17,603	80,694

Table 4: The sum of VaR(X) and VaR(Y) (line 1) versus VaR(X + Y) (line 2) for couple of distributions: LN = lognormal, WE = Weibull, GPD = Generalised Pareto, GH = Generalised Hyperbolic, AS = Alpha-Stable, GV = Generalised Extreme Value, GVB = Generalised Extreme Value calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th.

WE-WE 1	501	820	1,505	2,352	4,878	6,187	9,703
WE-WE 2	501	821	1,510	2,352	4,879	6,185	9,807
WE-GPD 1	4,376	6,259	11,498	21,082	86,961	161,051	680,774
WE-GPD 2	58,916	60,639	65,520	74,662	138,368	209,701	726,035
WE-GH 1	418	690	1,379	2,494	8,460	13,300	32,534
WE-GH 2	533	795	1,379	2,208	6,472	10,534	27,998
WE-AS 1	431	692	1,335	2,386	9,544	18,665	103,193
WE-AS 2	528	779	1,341	2,148	7,556	16,025	101,095
WE-GV 1	25,186	137,542	2,098,044	28,700,884	$10.73e9$	$134.51e9$	$47,029e9$
WE-GV 2	25,197	138,107	2,094,946	29,156,852	$10.47e9$	$135.38e9$	$45,501e9$
WE-GVb 1	420	692	1,379	2,504	9,616	17,805	82,315
WE-GVb 2	534	796	1,381	2,237	7,710	15,281	79,250

Table 5: The sum of VaR(X) and VaR(Y) (line 1) versus VaR(X + Y) (line 2) for couple of distributions: LN = lognormal, WE = Weibull, GPD = Generalised Pareto, GH = Generalised Hyperbolic, AS = Alpha-Stable, GV = Generalised Extreme Value, GVB = Generalised Extreme Value calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th.

GPD-GPD 1	8,250	11,699	21,490	39,812	169,044	315,915	1,351,846
GPD-GPD 2	117,080	120,546	130,394	148,749	276,271	418,831	1,452,006
GPD-GH 1	4,292	6,129	11,372	21,224	90,543	168,164	703,606
GPD-GH 2	59,005	60,888	66,096	75,538	139,002	209,869	726,103
GPD-AS 1	4,305	6,131	11,328	21,116	91,627	173,528	774,264
GPD-AS 2	58,987	60,890	66,273	76,314	147,644	229,984	834,971
GPD-GV 1	29,061	142,981	2,108,036	28,719,614	$10.73e9$	$134.51e9$	$47,029e9$
GPD-GV 2	92,215	210,767	2,181,852	29,254,626	$10.47e9$	$135.38e9$	$45,501e9$
GPD-GVb 1	4,292	6,129	11,372	21,224	90,543	168,164	703,606
GPD-GVb 2	59,005	60,888	66,096	75,538	139,002	209,869	726,103
GH-GH 1	335	559	1,253	2,635	12,043	20,413	55,366
GH-GH 2	335	559	1,253	2,635	12,043	20,413	55,366
GH-AS 1	348	562	1,209	2,527	13,126	25,778	126,024
GH-AS 2	442	683	1,393	2,778	12,596	23,446	104,497
GH-GV 1	25,103	137,412	2,097,918	28,701,025	$10.73e9$	$134.51e9$	$47,029e9$
GH-GV 2	25,635	138,429	2,095,206	29,157,735	$10.47e9$	$135.38e9$	$45,501e9$
GH-GVb 1	336	562	1,252	2,645	13,198	24,917	105,146
GH-GVb 2	446	703	1,451	2,895	12,502	22,224	84,680

Table 6: The sum of VaR(X) and VaR(Y) (line 1) versus VaR(X + Y) (line 2) for couple of distributions: LN = lognormal, WE = Weibull, GPD = Generalised Pareto, GH = Generalised Hyperbolic, AS = Alpha-Stable, GV = Generalised Extreme Value, GVB = Generalised Extreme Value calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th.

AS-AS 1	361	564	$1,165$	$2,419$	$14,210$	$31,142$	$196,682$
AS-AS 2	360	562	$1,159$	$2,428$	$14,153$	$31,459$	$201,447$
AS-GV 1	$25,116$	$137,414$	$2,097,873$	$28,700,918$	$10.73e9$	$134.51e9$	$47,029e9$
AS-GV 2	$26,139$	$140,091$	$2,099,977$	$29,175,188$	$10.47e9$	$135.38e9$	$45,501e9$
AS-GVb 1	349	564	$1,208$	$2,537$	$14,282$	$30,282$	$175,804$
AS-GVb 2	443	683	$1,399$	$2,849$	$15,645$	$33,285$	$189,589$
GV-GV 1	$49,871$	$274,264$	$4,194,582$	$57,399,416$	$21.46e9$	$269e9$	$94,058e9$
GV-GV 2	$49,844$	$275,821$	$4,189,583$	$58,313,419$	$20.94e9$	$271e9$	$91,002e9$
GV-GVb 1	$25,105$	$137,414$	$2,097,917$	$28,701,036$	$10.73e9$	$134.51e9$	$47,029e9$
GV-GVb 2	$26,105$	$139,855$	$2,099,195$	$29,174,309$	$10.47e9$	$135.38e9$	$45,501e9$
GVb-GVb 1	338	564	$1,252$	$2,656$	$14,353$	$29,422$	$154,927$
GVb-GVb 2	340	565	$1,251$	$2,663$	$14,609$	$29,967$	$158,273$

Table 7: The sum of VaR(X) and VaR(Y) (line 1) versus VaR(X + Y) (line 2) for couple of distributions: LN = lognormal, WE = Weibull, GPD = Generalised Pareto, GH = Generalised Hyperbolic, AS = Alpha-Stable, GV = Generalised Extreme Value, GVB = Generalised Extreme Value calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th.

LN-LN 1	$1,895$	$2,587$	$4,226$	$6,616$	$16,572$	$23,652$	$50,725$
LN-LN 2	$1,895$	$2,587$	$4,226$	$6,616$	$16,572$	$23,652$	$50,725$
LN-WE 1	$1,727$	$2,302$	$3,574$	$5,293$	$11,739$	$16,021$	$31,500$
LN-WE 2	$1,541$	$1,970$	$2,882$	$4,092$	$8,841$	$12,269$	$25,675$
LN-GPD 1	$87,496$	$101,478$	$140,329$	$211,059$	$682,080$	$1,191,608$	$4,513,150$
LN-GPD 2	$87,065$	$100,726$	$138,767$	$208,277$	$674,213$	$1,180,157$	$4,488,157$
LN-GH 1	$2,146$	$2,984$	$5,081$	$8,347$	$23,114$	$33,777$	$71,406$
LN-GH 2	$1,996$	$2,698$	$4,383$	$6,898$	$17,681$	$25,214$	$50,397$
LN-AS 1	$16,694$	$24,801$	$48,732$	$95,726$	$459,981$	$905,044$	$4,350,967$
LN-AS 2	$16,545$	$24,525$	$48,067$	$94,322$	$454,147$	$895,398$	$4,326,497$
LN-GV 1	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
LN-GV 2	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
LN-GVb 1	$5,608$	$8,174$	$15,460$	$29,105$	$126,073$	$237,314$	$1,032,332$
LN-GVb 2	$5,457$	$7,888$	$14,762$	$27,640$	$120,229$	$227,765$	$1,008,148$

Table 8: The sum of ES(X) and ES(Y) (line 1) versus ES(X + Y) (line 2) for couple of distributions: LN = lognormal, WE = Weibull, GPD = Generalised Pareto, GH = Generalised Hyperbolic, AS = Alpha-Stable, GV = Generalised Extreme Value, GVB = Generalised Extreme Value calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th.

WE-WE 1	$1,559$	$2,016$	$2,921$	$3,970$	$6,905$	$8,390$	$12,276$
WE-WE 2	$1,559$	$2,016$	$2,921$	$3,970$	$6,905$	$8,390$	$12,276$
WE-GPD 1	$87,328$	$101,193$	$139,676$	$209,736$	$677,247$	$1,183,977$	$4,493,926$
WE-GPD 2	$86,887$	$100,505$	$138,515$	$208,044$	$674,087$	$1,180,072$	$4,488,101$
WE-GH 1	$1,978$	$2,698$	$4,428$	$7,024$	$18,280$	$26,146$	$52,182$
WE-GH 2	$1,810$	$2,389$	$3,739$	$5,758$	$15,192$	$22,257$	$46,312$
WE-AS 1	$16,526$	$24,516$	$48,079$	$94,403$	$455,148$	$897,413$	$4,331,742$
WE-AS 2	$16,359$	$24,217$	$47,423$	$93,172$	$452,023$	$893,523$	$4,325,897$
WE-GV 1	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2447e18$
WE-GV 2	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2447e18$
WE-GVb 1	$5,440$	$7,889$	$14,807$	$27,782$	$121,240$	$229,683$	$1,013,108$
WE-GVb 2	$5,270$	$7,579$	$14,119$	$26,506$	$118,106$	$225,770$	$1,007,256$

Table 9: The sum of ES(X) and ES(Y) (line 1) versus ES(X + Y) (line 2) for couple of distributions: LN = lognormal, WE = Weibull, GPD = Generalised Pareto, GH = Generalised Hyperbolic, AS = Alpha-Stable, GV = Generalised Extreme Value, GVB = Generalised Extreme Value calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th.

GPD-GPD 1	173,097	200,369	276,431	415,503	1,347,588	2,359,564	8,975,575
GPD-GPD 2	173,097	200,369	276,431	415,503	1,347,588	2,359,564	8,975,575
GPD-GH 1	87,747	101,874	141,183	212,791	688,622	1,201,732	4,533,832
GPD-GH 2	87,330	101,092	139,298	208,887	674,421	1,180,208	4,488,112
GPD-AS 1	102,295	123,692	184,834	300,169	1,125,489	2,073,000	8,813,392
GPD-AS 2	101,891	122,938	182,933	295,782	1,098,582	2,016,042	8,499,442
GPD-GV 1	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
GPD-GV 2	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
GPD-GVb 1	91,209	107,065	151,562	233,548	791,581	1,405,270	5,494,758
GPD-GVb 2	90,787	106,267	149,558	229,042	766,781	1,355,085	5,243,081
GH-GH 1	2,397	3,380	5,935	10,078	29,655	43,901	92,088
GH-GH 2	2,397	3,380	5,935	10,078	29,655	43,901	92,088
GH-AS 1	16,945	25,197	49,586	97,457	466,523	915,168	4,371,648
GH-AS 2	16,809	24,941	48,924	95,926	458,199	899,741	4,327,096
GH-GV 1	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
GH-GV 2	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
GH-GVb 1	5,858	8,571	16,314	30,836	132,615	247,439	1,053,014
GH-GVb 2	5,722	8,305	15,616	29,227	124,294	232,340	1,009,422

Table 10: The sum of ES(X) and ES(Y) (line 1) versus ES(X + Y) (line 2) for couple of distributions: LN = lognormal, WE = Weibull, GPD = Generalised Pareto, GH = Generalised Hyperbolic, AS = Alpha-Stable, GV = Generalised Extreme Value, GVB = Generalised Extreme Value calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th.

AS-AS 1	$31,493$	$47,015$	$93,237$	$184,836$	$903,390$	$1,786,436$	$8,651,209$
AS-AS 2	$31,493$	$47,015$	$93,237$	$184,836$	$903,390$	$1,786,436$	$8,651,209$
AS-GV 1	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
AS-GV 2	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
AS-GVb 1	$20,406$	$30,388$	$59,965$	$118,215$	$569,482$	$1,118,706$	$5,332,574$
AS-GVb 2	$20,270$	$30,130$	$59,302$	$116,655$	$559,704$	$1,097,691$	$5,212,237$
GV-GV 1	$16e18$	$24e18$	$48e18$	$97e18$	$489e18$	$979e18$	$4,895e18$
GV-GV 2	$16e18$	$24e18$	$48e18$	$97e18$	$489e18$	$979e18$	$4,895e18$
GV-GVb 1	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
GV-GVb 2	$8e18$	$12e18$	$24e18$	$48e18$	$244e18$	$489e18$	$2,447e18$
GVb-GVb 1	$9,320$	$13,761$	$26,693$	$51,593$	$235,574$	$450,977$	$2,013,940$
GVb-GVb 2	$9,320$	$13,761$	$26,693$	$51,593$	$235,574$	$450,977$	$2,013,940$

Table 11: The sum of ES(X) and ES(Y) (line 1) versus ES(X + Y) (line 2) for couple of distributions: LN = lognormal, WE = Weibull, GPD = Generalised Pareto, GH = Generalised Hyperbolic, AS = Alpha-Stable, GV = Generalised Extreme Value, GVB = Generalised Extreme Value calibrated on maxima. The percentiles represented are the 70th, 80th, 90th, 95th, 99th, 99.5th and 99.9th.

80.00%	81.00%	82.00%	83.00%	84.00%	85.00%	86.00%	87.00%
-81.843	-74.943	-66.539	-57.410	-47.461	-35.212	-20.496	-3.984
88.00%	89.00%	90.00%	91.00%	92.00%	93.00%	94.00%	95.00%
16.247	40.129	67.997	102.443	144.756	196.882	266.676	360.135
96.00%	97.00%	98.00%	99.00%	99.50%	99.90%	99.95%	99.99%
489.356	677.618	1,011.196	1,696.400	2,581.672	4,858.396	5,761.766	10,964.930

Table 12: This table shows the differences between the sum VaR(X) and the VaR(Y) and the VaR(X + Y). The random variable X has been generated using a Weibull and Y has been obtained from a lognormal distribution. When the values are positive, the VaR is sub-additive, when the values are negative the VaR is not. The turning points are highlighted in bold.

80.00%	81.00%	82.00%	83.00%	84.00%	85.00%	86.00%	87.00%
-86.104	-82.891	-80.004	-75.764	-69.887	-63.385	-55.082	-45.380
88.00%	89.00%	90.00%	**91.00%**	92.00%	93.00%	94.00%	95.00%
-34.810	-21.030	-2.510	**23.340**	54.970	99.660	159.200	249.830
96.00%	97.00%	98.00%	99.00%	99.50%	99.90%	99.95%	**99.99%**
393.730	632.630	1,098.500	2,170.800	3,052.900	4,784.190	17,905.440	**−633,422.500**

Table 13: This table shows the differences between the sum VaR(X) and the VaR(Y) and the VaR(X + Y). The random variable X has been generated using a Weibull and Y has been obtained from an Alpha-stable distribution. When the values are positive, the VaR is sub-additive, when the values are negative the VaR is not. The turning points are highlighted in bold.

-0.012	0.022	-0.013	-0.018	-0.015	-0.031	-0.020	-0.026
-0.038	**0.011**	0.028	0.023	0.022	0.024	0.044	0.073
0.074	0.080	0.139	0.144	0.194	0.171	0.167	0.163
0.142	0.141	0.134	0.150	0.179	0.175	0.105	0.107
0.016	**-0.001**	-0.002	-0.003	0.013	-0.021	-0.048	-0.011
-0.016	**0.045**	0.074	0.032	0.074	0.166	0.124	0.104
0.098	0.019	**-0.037**	-0.079	-0.100	-0.120	-0.144	-0.047
-0.070	-0.086	-0.136	-0.234	-0.291	-0.352	-0.272	-0.197
-0.098	0.038	0.121	-0.313	-0.299	-0.483	-0.621	-0.422
-0.457	**0.099**	0.272	0.381	0.430	0.656	0.754	0.533
0.693	1.035	0.715	1.087	0.778	$\mathbf{-0.167}$	-0.479	-0.522
-0.759	-3.391	-2.265	-4.190	-3.137	-6.484	-1.975	**9.502**
6.873	16.636	69.495	50.091	$7,118.689$	$8,798.144$	$\mathbf{-148,979.500}$	NA

Table 14: This table shows the differences between the sum VaR(X) and the VaR(Y) and the VaR(X + Y). The random variable X and Y have been obtained on 2 identical GEV distribution. When the values are positive, the VaR is sub-additive, when the values are negative the VaR is not. The turning points are highlighted in bold. The percentiles represented are sequentially going form 1% to 99% by 1%, and to capture the tail, the 99.95th, 99.9th, 99.95th and 99.99th percentiles are added.